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A Time-Domain Finite Element Method for Helmholtz Equations

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A time-domain finite element method is developed to approximate the electromagnetic fields scattered by a bounded, inhomogeneous two-dimensional cavity embedded in the infinite ground plane. The time-dependent scattering problem is first discretized in time by Newmark's time-stepping scheme. A nonlocal boundary condition on the cavity aperture is constructed to reduce the computational domain to the cavity itself. The variational problems using finite element methods are shown to have unique solutions. Numerical experiments for both TE and TM polarizations demonstrate the accuracy and stability of the method.

Key Words: Helmholtz equations; finite element methods; Newmark scheme; nonreflecting boundary condition; scattering.

1. INTRODUCTION

Time-domain methods for electromagnetic scattering problems have enjoyed growing popularity in recent years due to their abilities to generate wide-band data and the advances in computer technology. Since most scattering problems are defined in an infinite domain, one is faced with the problem of truncating the infinite domain to a finite computational domain without introducing excessive error. This can be accomplished by introducing an artificial boundary enclosing the scattering body and an appropriate boundary condition on it. In [6–8], the artificial boundary is chosen to be a circle or a sphere, on which an exact nonlocal boundary condition (also called nonreflecting boundary condition) is constructed to couple the field's exterior to the artificial boundary to those in the interior of the boundary. The exterior fields can be found analytically by using Fourier series or spherical harmonic series. The interior fields are numerically computed by a finite element method. If the scattering body is thin and long in one direction, the circular or spherical boundaries are not efficient due to the unnecessary triangulations inside the circle or sphere. For such an elongated scattering body, it is more desirable to place the artificial boundary as close to the

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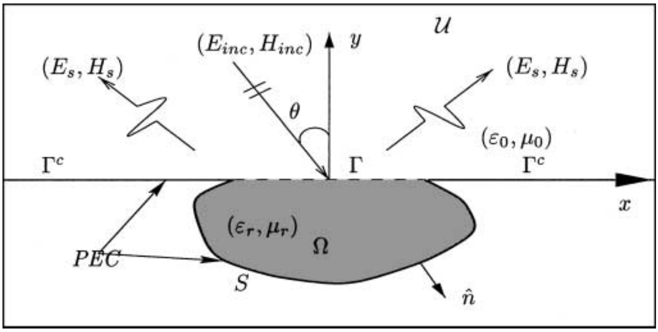


FIG. 1. Cavity setting.

body as possible. In this case, absorbing boundary conditions (ABC) may be used as in [11, 12]. For scattering problems in a cavity, one can place an artificial boundary at the cavity opening, which is a line segment in 2-D or a planar region in 3-D. In frequency-domain cavity problems, exact boundary conditions are constructed by using Green functions [10] or Fourier transforms [2, 13]. In this paper, we consider the time-domain scattering problems in two-dimensional cavities. We reduce the infinite problem domain to a finite computational domain by introducing an exact nonlocal boundary condition onto the cavity aperture. We use the finite element method to solve the problem in the cavity interior. Denote Ω as a bounded 2-D cavity embedded in an infinite ground plane, where Ω is a bounded Lipschitz continuous region in $\mathbb{R}^2$ such that

$$\Omega \subset \{\boldsymbol{r} = (x, y) \in \mathbb{R}^2 : y < 0\}, \quad \bar{\Omega} \cap \{\boldsymbol{r} = (x, y) \in \mathbb{R}^2 : y = 0\} \neq \emptyset.$$

We denote S as the cavity walls, Γ as the cavity aperture, $\Gamma^c = \{\boldsymbol{r} = (x, y) \in \mathbb{R}^2 : y = 0\} \backslash \Gamma$, and $\mathcal{U} = \{\boldsymbol{r} = (x, y) : y > 0\}$ as the upper half plane. We assume that the ground is perfectly electric conducting (PEC). Ω is either empty with $\varepsilon_r = \varepsilon_0 = 1$ or filled with material whose relative permittivity is $\varepsilon_r > 1$ (see Fig. 1). We assume that cavities are nonmagnetic; that is, $\mu_r = \mu_0 = 1$. The scattering problem we consider in this paper is that given an incident electromagnetic field $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ interacting with the cavity Ω to produce a total field $(\boldsymbol{E}, \boldsymbol{H})$, determine the scattered fields $\boldsymbol{E}^s = \boldsymbol{E} - \boldsymbol{E}^i - \boldsymbol{E}^r$, $\boldsymbol{H}^s = \boldsymbol{H} - \boldsymbol{H}^i - \boldsymbol{H}^r$, where $\boldsymbol{E}^r$, $\boldsymbol{H}^r$ are the reflected electric and magnetic fields, respectively. The scattering problem can be decomposed into the two fundamental polarizations: transverse magnetic (TM) and transverse electric (TE). In the TM case, given the incident electric field of the form

$$\boldsymbol{E}^{inc} = \left(0, 0, E_z^{inc}\right) = (0, 0, u^{inc}), \tag{1.1}$$

we wish to find the total field $\boldsymbol{E} = (0, 0, u)$ such that

$$\begin{cases} -\Delta u + \varepsilon_r \dfrac{\partial^2 u}{\partial t^2} = 0 & \text{in } \Omega \times (0, T), \\ u = u^{inc} + u^{ref} + u^s & \text{on } \Gamma \times (0, T), \\ u = 0 & \text{on } S \cup \Gamma^c \times (0, T), \end{cases} \tag{1.2}$$

with the initial conditions

$$u(0,\boldsymbol{r}) = u_0(\boldsymbol{r}), \quad \frac{\partial u}{\partial t}(0,\boldsymbol{r}) = u_1(\boldsymbol{r}).$$

The scattered field is defined in the upper half plane and satisfies

$$\begin{cases} -\Delta u^s + \dfrac{\partial^2 u^s}{\partial t^2} = 0 & \text{in } \mathcal{U} \times (0, T), \\ u^s = 0 & \text{on } \Gamma^c \times (0, T), \end{cases} \tag{1.3}$$

and the radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial}{\partial r} + \frac{\partial}{\partial t}\right) u^s(t,\boldsymbol{r}) = 0, \quad r = |\boldsymbol{r}|. \tag{1.4}$$

By Maxwell's equations, the magnetic field $\boldsymbol{H}$ can be found in terms of the solution u.

In the TE polarization, given the incident magnetic wave

$$\boldsymbol{H}^{inc} = \left(0, 0, \boldsymbol{H}_z^{inc}\right) = (0, 0, u^{inc}), \tag{1.5}$$

we wish to solve for the total magnetic field $\boldsymbol{H} = (0, 0, u)$ such that

$$\begin{cases} -\nabla \cdot \left(\varepsilon_r^{-1} \nabla u\right) + \dfrac{\partial^2 u}{\partial t^2} = 0 & \text{in } \Omega \times (0, T), \\ \dfrac{1}{\varepsilon_r} \dfrac{\partial u}{\partial y} = \dfrac{\partial u^{inc}}{\partial y} + \dfrac{\partial u^{ref}}{\partial y} + \dfrac{\partial u^s}{\partial y} & \text{on } \Gamma \times (0, T), \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } S \times (0, T), \end{cases} \tag{1.6}$$

with the initial conditions

$$u(0,\boldsymbol{r}) = u_0(\boldsymbol{r}), \quad \frac{\partial u}{\partial t}(0,\boldsymbol{r}) = u_1(\boldsymbol{r}).$$

The scattered field u^s defined in the upper half plane satisfies

$$\begin{cases} -\Delta u^s + \dfrac{\partial^2 u^s}{\partial t^2} = 0 & \text{in } \mathcal{U} \times (0, T), \\ \dfrac{\partial u^s}{\partial n} = -\dfrac{\partial u^{inc}}{\partial n} - \dfrac{\partial u^{ref}}{\partial n} & \text{on } \Gamma^c \times (0, T), \end{cases} \tag{1.7}$$

and the radiation condition (1.4). By Maxwell's equations, the electric field $\boldsymbol{E}$ can be found in terms of the solution u.

In time-dependent problems, an exact nonreflecting boundary condition (NRBC) is nonlocal both in time and space. The nonlocality of NRBC in time requires the entire history of the solution on the artificial boundary and can be very expensive for large problems. In [5], the numerical methods for solving three-dimensional acoustic and elastic wave problems in the unbounded domain are based on the Kirchhoff-type boundary condition which is nonlocal in both time and space. Since this boundary condition requires storing solution at the artificial boundary for the amount of time it takes a wave to propagate the domain, the numerical scheme can be expensive for large objects. Furthermore, this boundary condition is inherently three dimensional. The two-dimensional case is harder due to the nature of the 2-D Green functions. For the time-dependent electromagnetic scattering problem (Maxwell's equations), exact NRBCs are derived in [8] by using vector spherical harmonic series expansions (or multipole decomposition) for the spherical artificial boundary. As discussed previously, the computational domain might be unnecessarily large when one of the dimensions of the scatterer is much larger than others, for example a thin and long body. To alleviate this problem, we discretize the time variable first by using the Newmark scheme. At each time step, a spatially nonlocal boundary condition is constructed right at the cavity aperture Γ so that the computational domain is reduced to the cavity itself. With a time-marching scheme like Newmark, one is able to numerically solve a large cavity problem since the memory requirement does not increase as time goes on. Though the nonlocal boundary condition is only exact for the semidiscretized problem, it does yield very accurate and stable results, as seen in the numerical experiments in Section 4.

The paper is organized as follows. In Section 2, we discretize the TM and TE equations in time using the Newmark time-stepping scheme to obtain semidiscrete problems defined in an infinite space. At each time step we construct exact nonlocal boundary operators on the cavity opening to couple the fields in the exterior of the cavity to those inside. This coupling enables the semidiscrete problems to be reduced to the cavity itself; hence, finite element methods can be applied to approximate the solutions. In both cases, TM and TE, we prove that at each time step the weak formulations using the exact nonlocal boundary operators have a unique solutions. In Section 3, we discuss the implementation of the finite element method. In Section 4, we present some of numerical experiments that show the accuracy and temporal stability of the Newmark–finite element scheme.

2. SEMIDISCRETE PROBLEM

In this section, we first discretize the TM and TE equations in time by using the Newmark method [3, 14]. We then construct nonlocal boundary conditions for each time step. The boundary conditions couple the solutions in the exterior of the cavity to those in its interior, which enables the computational domain to be reduced to the cavity itself. The reduced partial differential equations will be numerically solved by the finite element methods in the next section.

Let $\mathcal{N} > 0$ be a positive integer. Define the constant time step $\Delta t = T/\mathcal{N}$. For each $n = 0, 1, \ldots, \mathcal{N}$, we denote $u^n(x)$ and $\dot{u}^n(x)$ as the temporal approximations of $u(\boldsymbol{r}, t_n)$ and $\frac{\partial u}{\partial t}(\boldsymbol{r}, t_n)$, where $t_n = n\Delta t$, $\boldsymbol{r} = (x, y) \in \mathbb{R}^2$. The Newmark method is defined by the expansions

$$\begin{aligned} u^{n+1} &= u^n + \Delta t \dot{u}^n + \frac{\Delta t^2}{2}[2\beta \ddot{u}^{n+1} + (1-2\beta)\ddot{u}^n], && 0 \le n \le \mathcal{N} - 1, \\ \dot{u}^{n+1} &= \dot{u}^n + \Delta t[\gamma \ddot{u}^{n+1} + (1-\gamma)\ddot{u}^n], && 0 \le n \le \mathcal{N} - 1, \end{aligned}$$

where $\ddot{u}^n$ satisfies, for the TM case,

$$-\Delta u^{n+1} + \varepsilon_r \ddot{u}^{n+1} = 0, \quad 0 \le n \le \mathcal{N} - 1,$$

or, for the TE case,

$$-\nabla\cdot\left(\varepsilon_r^{-1}\nabla u^{n+1}\right)+\ddot{u}^{n+1}=0,\quad 0\le n\le\mathcal{N}-1$$

and

$$u^0=u_0,\quad \dot{u}^0=u_1.$$

It is easy to see that the above time-stepping scheme is explicit when $\beta=0$ and that it is implicit otherwise. The scheme is of second-order accuracy when $\gamma=1/2$ and of first order otherwise. For convenience, we express the Newmark scheme in the predictor–corrector form as

Prediction.

$$\begin{aligned}\tilde{u}^{n+1}&=u^n+\Delta t\dot{u}^n+\frac{(\Delta t)^2}{2}(1-2\beta)\ddot{u}^n,\\ \tilde{\dot{u}}^{n+1}&=\dot{u}^n+\Delta t(1-\gamma)\ddot{u}^n.\end{aligned}\tag{2.1}$$

Solution.

$$(\mathrm{TM})\begin{cases}-\Delta u^{n+1}+\alpha^2\varepsilon_r u^{n+1}=\alpha^2\varepsilon_r\tilde{u}^{n+1} & \text{in }\Omega,\\ u^{n+1}=0 & \text{on }S,\\ u^{n+1}=u^{s,n+1} & \text{on }\Gamma(\text{since }u^{inc}+u^{ref}=0\text{ on }\Gamma),\end{cases}$$

$$(\mathrm{TE})\begin{cases}-\nabla\cdot\left(\nabla\varepsilon_r^{-1}u^{n+1}\right)+\alpha^2u^{n+1}=\alpha^2\varepsilon_r\tilde{u}^{n+1} & \text{in }\Omega,\\ \partial u^{n+1}/\partial n=0 & \text{on }S,\\ u^{n+1}=u^{inc,n+1}+u^{ref,n+1}+u^{s,n+1} & \text{on }\Gamma.\end{cases}$$

Correction.

$$\begin{aligned}\ddot{u}^{n+1}&=\alpha^2(u^{n+1}-\tilde{u}^{n+1}),\\ \dot{u}^{n+1}&=\tilde{\dot{u}}^{n+1}-\Delta t\gamma\ddot{u}^{n+1},\end{aligned}\tag{2.2}$$

where $\alpha^2=\frac{1}{\Delta t^2\beta}$.

2.1. *TM Polarization*

Here we solve the problem at each time step $t=t_{n+1}$ using finite element methods. In this case, the scattered field $u^{s,n+1}$ satisfies

$$\begin{cases}-\Delta u^{s,n+1}+\alpha^2u^{s,n+1}=\alpha^2\tilde{u}^{s,n+1} & \text{in }\mathcal{U},\\ u^{s,n+1}=g & \text{on }\Gamma,\\ u^{s,n+1}=0 & \text{on }\Gamma^c,\end{cases}\tag{2.3}$$

since $\varepsilon_r=1$ in $\mathcal{U}$. Here $g\stackrel{\text{def}}{=}u^{n+1}$ on Γ.

LEMMA 2.1. *Let $g \in H^{1/2}(\Gamma)$ be given. Then the exterior problem* (2.3) *has a unique solution*,

$$u^{s,n+1}(\boldsymbol{r}) = \alpha^2 \int_0^\infty \int_{-\infty}^\infty G_\alpha^e(\boldsymbol{r}, \boldsymbol{r}')\tilde{u}^{s,n+1}(\boldsymbol{r}')\, d\boldsymbol{r}' + \frac{1}{\pi}\int_\Gamma \frac{\partial}{\partial y} K_0(\alpha|\boldsymbol{r} - x'\hat{x}|) g(x')\, dx', \tag{2.4}$$

where G_α^e is the modified Green function defined by

$$G_\alpha^e(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{2\pi}[K_0(\alpha|\boldsymbol{r} - \boldsymbol{r}'|) - K_0(\alpha|\boldsymbol{r} - \boldsymbol{r}_i'|)], \boldsymbol{r}_i' = (x', -y').$$

Proof. Note that G_α^e satisfies the Dirichlet problem [9]

$$\begin{cases} -\Delta G_\alpha^e(\boldsymbol{r}, \boldsymbol{r}') + \alpha^2 G_\alpha^e(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}') & \text{in } \mathcal{U}, \\ G_\alpha^e = 0 & \text{on } \{y' = 0\} \text{ or } \{y = 0\}. \end{cases}$$

Hence, the solution $u^{s,n+1}$ to (2.3) can be found analytically as

$$\begin{aligned} u^{s,n+1}(\boldsymbol{r}) &= \alpha^2 \int_{\mathcal{U}} G_\alpha^e(\boldsymbol{r}, \boldsymbol{r}')\tilde{u}^{s,n+1}(\boldsymbol{r}')\, d\boldsymbol{r}' + \int_\Gamma \frac{\partial G_\alpha^e}{\partial n'}(\boldsymbol{r}, x'\hat{x}) g(x')\, dx' \\ &= \alpha^2 \int_0^\infty \int_{-\infty}^\infty G_\alpha^e(\boldsymbol{r}, \boldsymbol{r}')\tilde{u}^{s,n+1}(\boldsymbol{r}')\, d\boldsymbol{r}' - \int_\gamma \frac{\partial G_\alpha^e}{\partial y'}(\boldsymbol{r}; x'\hat{x}) g(x')\, dx', \end{aligned}$$

where $\boldsymbol{r} \in \mathcal{U}$. Here, $\hat{n}'$ is the outward normal on Γ for the primed variables. It is clear that $\hat{n}' = -\hat{y}$ on Γ. By a direct computation, we see that

$$u^{s,n+1}(\boldsymbol{r}) = \alpha^2 \int_0^\infty \int_{-\infty}^\infty G_\alpha^e(\boldsymbol{r}, \boldsymbol{r}')\tilde{u}^{s,n+1}(\boldsymbol{r}')\, d\boldsymbol{r}' + \frac{1}{\pi}\int_\Gamma \frac{\partial}{\partial y} K_0(\alpha|\boldsymbol{r} - x'\hat{x}|) g(x')\, dx'.$$

Hence, (2.4) yields the analytical solution to the exterior problem (2.3).

Taking the partial derivative of $u^{s,n+1}$ with respect to y yields

$$\frac{\partial u^{s,n+1}(\boldsymbol{r})}{\partial y} = \alpha^2 \int_0^\infty \int_{-\infty}^\infty \frac{\partial G_\alpha^e}{\partial y}(\boldsymbol{r}, \boldsymbol{r}')\tilde{u}^{s,n+1}(\boldsymbol{r}')\, d\boldsymbol{r}' + \frac{1}{\pi}\int_\Gamma \frac{\partial^2}{\partial y^2} K_0(\alpha|\boldsymbol{r} - x'\hat{x}|) g(x')\, dx'.$$

However, we also note that

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) K_0(\alpha|\boldsymbol{r} - x'\hat{x}|) + \alpha^2 K_0(\alpha|\boldsymbol{r} - x'\hat{x}|) = 0, \quad y > 0.$$

So,

$$\begin{aligned} \frac{\partial u^{s,n+1}}{\partial y} &= \alpha^2 \int_0^\infty \int_{-\infty}^\infty \frac{\partial G_\alpha^e}{\partial y}(\boldsymbol{r}, \boldsymbol{r}')\tilde{u}^{s,n+1}(\boldsymbol{r}')\, d\boldsymbol{r}' \\ &\quad + \frac{1}{\pi}\int_\Gamma \left(-\alpha^2 + \frac{\partial^2}{\partial x^2}\right) K_0(\alpha|\boldsymbol{r} - x'\hat{x}|) g(x')\, dx'. \end{aligned}$$

By taking the limit $y \to 0$, we obtain

$$\left.\frac{\partial u^{s,n+1}}{\partial y}\right|_{y=0} = \tilde{H}^{n+1}(x) + T^e_\alpha g(x), \tag{2.5}$$

where

$$\tilde{H}^{n+1}(x) \overset{\text{def}}{=} \alpha^2 \int_0^\infty \int_{-\infty}^\infty \frac{\partial G^e_\alpha}{\partial y}(x, 0; \boldsymbol{r}')\tilde{u}^{s,n+1}(\boldsymbol{r}')\, d\boldsymbol{r}' \tag{2.6}$$

and the nonlocal boundary operator $T^e_\alpha : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is defined by

$$T^e_\alpha g(x) \overset{\text{def}}{=} \frac{1}{\pi} \int_\Gamma \left(-\alpha^2 + \frac{\partial^2}{\partial x^2}\right) K_0(\alpha|x - x'|)g(x')\, dx'. \tag{2.7}$$

Define $V = \{v \in H^1(\Omega) : u = 0 \text{ on } S\}$. From (2.5) we get

$$\frac{\partial u^{n+1}}{\partial y} = \frac{\partial u^{inc,n+1}}{\partial y} + \frac{\partial u^{ref,n+1}}{\partial y} + \tilde{H}^{n+1} + T^e_\alpha u^{s,n+1} \quad \text{on } \Gamma.$$

Since $u^{inc,n+1} + u^{ref,n+1} = 0$ on Γ, we also have $u^{s,n+1} = u^{n+1}$ on Γ. Hence, we have

$$\frac{\partial u^{n+1}}{\partial y} = 2\frac{\partial u^{inc,n+1}}{\partial y} + \tilde{H}^{n+1} + T^e_\alpha u^{n+1} \quad \text{on } \Gamma. \tag{2.8}$$

This is the nonlocal boundary condition imposed on the cavity opening, and the variational formulation is

$$\int_\Omega \{\nabla u^{n+1} \cdot \nabla v + \alpha^2 u^{n+1} v\}\, dx - \int_\Gamma T^e_\alpha u^{n+1}|_\Gamma v\, dx$$
$$= \alpha^2 \int_\Omega \varepsilon_r \tilde{u}^{n+1} v\, dx + \int_\Gamma \tilde{H}^{n+1} v\, dx + 2\int_\Gamma \frac{\partial u^{inc,n+1}}{\partial y} v\, dx.$$

Therefore, we wish to solve the variational problem

$$a(u^{n+1}, v) = b^{n+1}(v), \tag{2.9}$$

where

$$a(u, v) = (\nabla u, \nabla v) + \alpha^2(\varepsilon_r u, v) - \left\langle T^e_\alpha u, v\right\rangle_\Gamma, \tag{2.10}$$

$$b^{n+1}(v) = \alpha^2(\varepsilon_r \tilde{u}^{n+1}, v) + \langle \tilde{H}^{n+1}, v\rangle_\Gamma + 2\left\langle \frac{\partial u^{inc,n+1}}{\partial y}, v\right\rangle_\Gamma. \tag{2.11}$$

The following is the main result of the section.

THEOREM 2.2. *Let $V \subset L^2(\Omega)$ be defined as*

$$V = \{u \in H^1(\Omega) : u = 0 \quad \text{on } S\}.$$

Then the variational problem (2.9) *has a unique solution in V.*

2.2. *TE Polarization*

Consider the TE equation (1.6). To derive the exact boundary condition on Γ we define the function

$$z = u - u^{inc} + u^{ref}.$$

By a direct computation, it is clear that

$$\frac{\partial z}{\partial y} = 0 \quad \text{on } \{y = 0\}.$$

Then in Ω, z satisfies

$$\begin{cases} -\nabla \cdot \left(\varepsilon_r^{-1} z\right) + \dfrac{\partial^2 z}{\partial t^2} = f & \text{in } \Omega \times (0, T), \\ \dfrac{1}{\varepsilon_r} \dfrac{\partial z}{\partial y}\bigg|_{y=0^-} = \dfrac{\partial z}{\partial y}\bigg|_{y=0^+} & \text{on } \Gamma \times (0, T), \\ \dfrac{\partial z}{\partial n} = -\dfrac{\partial u^{inc}}{\partial n} + \dfrac{\partial u^{ref}}{\partial n} & \text{on } S \times (0, T), \end{cases} \tag{2.12}$$

where

$$f = -\nabla \cdot \left(\varepsilon_r^{-1}(-u^{inc} + u^{ref})\right) + \frac{\partial^2(-u^{inc} + u^{ref})}{\partial t^2}.$$

In the upper half plane $\mathcal{U}$, z satisfies

$$\begin{cases} -\Delta z + \dfrac{\partial^2 z}{\partial t^2} = 0 & \text{in } \mathcal{U} \times (0, T), \\ \dfrac{\partial z}{\partial y} = 0 & \text{on } \Gamma^c \times (0, T). \end{cases}$$

To solve for u, it is sufficient to solve for z. So, we now concentrate on (2.12). The predictor–corrector scheme of the TE problem is easily obtained in terms of z^{n+1} by replacing u^{n+1} in (2.1) and (2.2) with z^{n+1} and the equation in the solution part with

$$\begin{cases} -\nabla \cdot \left(\varepsilon_r^{-1} \nabla z^{n+1}\right) + \alpha^2 z^{n+1} = \alpha^2 \tilde{z}^{n+1} + f^{n+1} & \text{in } \Omega, \\ \dfrac{\partial z^{n+1}}{\partial n} = -\dfrac{\partial u^{inc,n+1}}{\partial n} + \dfrac{\partial u^{ref,n+1}}{\partial n} & \text{on } S, \\ z^{n+1} \text{ is continuous} & \text{on } \Gamma. \end{cases}$$

In the upper half space $\mathcal{U}$, z^{n+1} satisfies

$$\begin{cases} -\Delta z^{n+1} + \alpha^2 z^{n+1} = \alpha^2 \tilde{z}^{n+1} & \text{on } \mathcal{U}, \\ \dfrac{\partial z^{n+1}}{\partial n} = -g^{n+1}(x) & \text{on } \Gamma, \\ \dfrac{\partial z^{n+1}}{\partial y} = 0 & \text{on } \Gamma^c, \end{cases} \tag{2.13}$$

where

$$g^{n+1}(x) = \varepsilon_r^{-1}\frac{\partial z^{n+1}}{\partial y}.$$

LEMMA 2.3. *For each $g^{n+1} \in H^{-1/2}(\Gamma)$, there exists a unique solution z^{n+1} to* (2.13) *and is of the form*

$$z^{n+1}(\boldsymbol{r}) = \alpha^2 \int_{\mathcal{U}} G_\alpha^m(\boldsymbol{r}, \boldsymbol{r}')\tilde{z}^{n+1}(\boldsymbol{r}')\, d\boldsymbol{r}' - \int_\Gamma G_\alpha^m(\boldsymbol{r}, x'\hat{x})g^{n+1}(x')\, dx', \quad \boldsymbol{r} \in \mathcal{U}, \tag{2.14}$$

where G_α^m is the modified "magnetic" Green function defined by

$$G_\alpha^m(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{2\pi}[K_0(\alpha|\boldsymbol{r} - \boldsymbol{r}'|) + K_0(\alpha|\boldsymbol{r} - \boldsymbol{r}_i'|)]. \tag{2.15}$$

The proof is similar to that of Lemma 2.1. Thus, on Γ we have

$$z^{n+1}|_{y=0^+} = \tilde{H}^{n+1}(x) + T_\alpha^m g^{n+1}(x), \tag{2.16}$$

where

$$\tilde{H}^{n+1}(x) = \frac{\alpha^2}{\pi}\int_{\mathcal{U}} K_0(\alpha|(x - x')\hat{x} + y'\hat{y}|)\tilde{z}^{n+1}(x', y')\, dx'\, dy' \tag{2.17}$$

and T_α^m is a nonlocal boundary operator $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ defined by

$$T_\alpha^m \psi(x) = -\frac{1}{\pi}\int_\Gamma K_0(\alpha|x - x'|)\psi(x')\, dx', \quad \forall \psi \in H^{-1/2}(\Gamma). \tag{2.18}$$

By Proposition 2.5, $T_\alpha^m : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded,

$$\left\|T_\alpha^m \psi\right\|_{H^{1/2}(\Gamma)} \le C\|\psi\|_{H^{-1/2}(\Gamma)}, \quad \forall \psi \in H^{-1/2}(\Gamma).$$

Therefore, we obtain

$$\begin{cases} -\nabla \cdot \left(\varepsilon_r^{-1}\nabla z^{n+1}\right) + \alpha^2 z^{n+1} = \alpha^2 \tilde{z}^{n+1} + f^{n+1} & \text{in } \Omega, \\ z^{n+1} = \tilde{H}^{n+1} + T_\alpha^m \left(\left.\dfrac{\partial z^{n+1}}{\partial y}\right|_{y=0+}\right) & \text{on } \Gamma, \\ \dfrac{\partial z^{n+1}}{\partial n} = -\dfrac{\partial u^{inc,n+1}}{\partial n} + \dfrac{\partial u^{ref,n+1}}{\partial n} & \text{on } S. \end{cases} \tag{2.19}$$

Now we set

$$w^{n+1} = z^{n+1} - \alpha^2 \int_{\mathcal{U}} G_\alpha^m(\boldsymbol{r}, \boldsymbol{r}')\tilde{z}^{n+1}(\boldsymbol{r}')\, d\boldsymbol{r}' \stackrel{\text{def}}{=} z^{n+1} - \tilde{\mathcal{H}}^{n+1}.$$

Note that

$$\begin{aligned} w^{n+1}|_\Gamma &= z^{n+1}|_\Gamma - \tilde{H}^{n+1} && \text{on } \mathcal{U}, \\ \frac{\partial w^{n+1}}{\partial y} &= \frac{\partial z^{n+1}}{\partial y} && \text{on } \Gamma. \end{aligned}$$

Thus, w^{n+1} satisfies

$$\begin{cases} -\nabla \cdot \left(\varepsilon_r^{-1} \nabla w^{n+1}\right) + \alpha^2 w^{n+1} = h^{n+1} & \text{in } \Omega, \\ w^{n+1} = T_\alpha^m \left(\dfrac{\partial w^{n+1}}{\partial y} \right) & \text{on } \Gamma, \\ \dfrac{\partial w^{n+1}}{\partial n} = -\dfrac{\partial u^{inc,n+1}}{\partial n} + \dfrac{\partial u^{ref,n+1}}{\partial n} - \dfrac{\partial \tilde{\mathcal{H}}^{n+1}}{\partial n} & \text{on } S, \end{cases} \tag{2.20}$$

where

$$h^{n+1} = \alpha^2 \tilde{z}^{n+1} + f^{n+1} - \left(-\nabla \cdot \left(\varepsilon_r^{-1} \nabla \tilde{\mathcal{H}}^{n+1}\right) + \alpha^2 \tilde{\mathcal{H}}^{n+1}\right).$$

We introduce the functional space

$$V = \left\{ v \in H^1(\Omega) : v = T_\alpha^m \left(\frac{\partial v}{\partial y} \right) \text{ on } \Gamma \right\}.$$

Then the variational form of (2.20) is to find $w^{n+1} \in V$ such that

$$a(w^{n+1}, v) = b^{n+1}(v), \quad \forall v \in V,$$

where

$$a(w^{n+1}, v) = \int_\Omega \varepsilon_r^{-1} \nabla w^{n+1} \cdot \nabla v + \alpha^2 w^{n+1} v - \int_\Gamma \frac{\partial w^{n+1}}{\partial y} T_\alpha^m \left(\frac{\partial v}{\partial y} \right)$$

and

$$b^{n+1}(v) = \int_\Omega h^{n+1} v + \int_\Gamma \left\{ -\frac{\partial u^{inc,n+1}}{\partial n} + \frac{\partial u^{ref,n+1}}{\partial n} - \frac{\partial \tilde{\mathcal{H}}^{n+1}}{\partial n} \right\} v.$$

THEOREM 2.4. *Let $V \subset L^2(\Omega)$ be defined by*

$$V = \left\{ v \in H^1(\Omega) : v = T_\alpha^m \frac{\partial v}{\partial y} \text{ on } \Gamma \right\}.$$

The variational problem of (2.20) *has a unique solution.*

In terms of u^{n+1}, (2.16) yields the nonlocal boundary

$$u^{n+1}|_{y=0^+} = 2u^{inc,n+1} + \tilde{H}^{n+1}(x) + T_\alpha^m g^{n+1}(x), \tag{2.21}$$

since

$$g^{n+1} = \frac{1}{\varepsilon_r} \frac{\partial z^{n+1}}{\partial y}\bigg|_{y=0^-} = \frac{\partial u^{n+1}}{\partial y}\bigg|_{y=0^+}.$$

Thus, with the relation in (2.21) the interior problem for TE polarization can be written as

$$\begin{cases} -\nabla \cdot \left(\varepsilon_r^{-1} \nabla u^{n+1}\right) + \alpha^2 u^{n+1} = \alpha^2 \tilde{u}^{n+1} & \text{in } \Omega, \\ u^{n+1}|_{y=0^-} = 2u^{inc,n+1} + \tilde{H}^{n+1} + T_\alpha^m \left(\dfrac{\partial u^{n+1}}{\partial y}\bigg|_{y=0^+} \right) & \text{on } \Gamma, \\ \dfrac{\partial u^{n+1}}{\partial n} = 0 & \text{on } S. \end{cases} \tag{2.22}$$

To prove Theorems 2.2 and 2.4, we need the following properties of pseudodifferential operators. (The proof is long and rather technical and is omitted here for brevity. Interested readers are referred to [4]).

PROPOSITION 2.5. *Let S_1 and S_{-1} be nonlocal operators on Γ defined by*

$$S_1(\phi)(x) = \gamma_0 r^+\left(\int_\Gamma \frac{\partial^2}{\partial y^2} K_0(\alpha|\boldsymbol{r} - x'|)\phi(x')\,dx'\right), \quad x \in \Gamma,$$
$$S_{-1}(\phi)(x) = \gamma_0 r^+\left(\int_\Gamma K_0(\alpha|\boldsymbol{r} - x'|)\phi(x')\,dx'\right), \qquad x \in \Gamma,$$

respectively, where $\phi \in H^s(\Gamma)$, $s \in \mathbb{R}$, r^+ is the restriction to $\mathbb{R}^2_+$, and γ_0 is the trace operator $\gamma_0 u = u|_\Gamma$. Then S_1 is a pseudodifferential operator of order 1 and S_{-1} is a pseudodifferential operator of order -1, which implies that

$$S_1 : H^s(\Gamma) \to H^{s-1}(\Gamma) \quad \text{and} \quad S_{-1} : H^s(\Gamma) \to H^{s+1}(\Gamma)$$

are bounded.

Therefore, we have

PROPOSITION 2.6. *The operator T^e_α defined in the TM problem is a pseudodifferential operator of order 1 and $T^e_\alpha : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is bounded. Similarly, the operator T^m_α defined in the TE problem is a pseudodifferential operator of order -1 and $T^m_\alpha : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded.*

Moreover, $\langle T^e_\alpha g, g\rangle_\Gamma$ is nonpositive for any $g \in H^{1/2}(\Gamma)$ and $\langle T^m_\alpha h, h\rangle_\Gamma$ is nonpositive for any $h \in H^{-1/2}(\Gamma)$.

Proof. Since T^e_α is defined by

$$\begin{aligned} T^e_\alpha g &= \frac{1}{\pi}\int_\Gamma \left(-\alpha^2 + \frac{\partial^2}{\partial x^2}\right) K_0(\alpha|x - x'|)g(x')\,dx' \\ &= \frac{1}{\pi}\int_\Gamma \left.\frac{\partial^2}{\partial y^2}\right|_{y=0} K_0(\alpha|\boldsymbol{r} - x'\hat{x}|)g(x')\,dx', \end{aligned}$$

it is the operator S_1 in Proposition 2.5. We can express $\langle T^e_\alpha g, g\rangle_\Gamma$ as

$$\begin{aligned} \left\langle T^e_\alpha g, g\right\rangle_\Gamma = &-\frac{\alpha^2}{\pi}\int_\Gamma dx\,\bar{g}(x)\int_\Gamma K_0(\alpha|x - x'|)g(x')\,dx' \\ &-\frac{1}{\pi}\int_\Gamma dx\frac{d\bar{g}(x)}{dx}\int_\Gamma K_0(\alpha|x - x'|)\frac{dg(x')}{dx'}dx'. \end{aligned} \tag{2.23}$$

The above identity is a direct consequence of the Green formula applied to (2.7). The symmetry of T^e_α follows directly (2.23). Let $\tilde{\Gamma}$ be a smooth closed curve such that $\Gamma \subset \tilde{\Gamma}$ and $\tilde{g} \in H^{1/2}(\tilde{\Gamma})$. We now set

$$w(x) = \int_{\tilde{\Gamma}} K_0(\alpha|x - x'|)\tilde{g}(x')\,dx', \quad x \in \mathbb{R}^2, \quad x' \in \tilde{\Gamma}. \tag{2.24}$$

Then we can easily show that w satisfies the equation

$$\begin{cases} -\Delta w + \alpha^2 w = 0 & \text{in } \mathbb{R}^2 \backslash \tilde{\Gamma}, \\ \left[\dfrac{\partial w}{\partial n}\right] = \tilde{g} & \text{on } \tilde{\Gamma}, \\ [w] = 0 & \text{on } \tilde{\Gamma}. \end{cases}$$

Thus, by the Green formula we get

$$\begin{aligned} 0 \le \int_{\mathbb{R}^2} |\nabla w|^2 + \alpha^2 |w|^2 &= \int_{\tilde{\Gamma}} w_i \frac{\partial \bar{w}_i}{\partial n} - w_e \frac{\partial \bar{w}_e}{\partial n} \, dx \\ &= \int_{\tilde{\Gamma}} w \left[\frac{\partial \bar{w}}{\partial n}\right] dx = \int_{\tilde{\Gamma}} w \bar{\tilde{g}} \, dx. \end{aligned}$$

Hence, by multiplying (2.24) with $\bar{\tilde{g}}$ and integrating the result over $\tilde{\Gamma}$, we obtain

$$0 \le \int_{\tilde{\Gamma}} w \bar{\tilde{g}} \, dx = \int_{\Gamma} dx \, \bar{g}(x) \int_{\Gamma} K_0(\alpha |x - x'|) g(x') \, dx'.$$

Therefore,

$$\left\langle T_\alpha^e g, g \right\rangle = -\langle K_0(\alpha |x - x'|) g, g \rangle - \langle K_0(\alpha |x - x'|) g', g' \rangle \le 0.$$

Similarly, since the operator T_α^m is defined by

$$T_\alpha^m h(x) = -\frac{1}{\pi} \int_{\Gamma} K_0(\alpha |x - x'|) h(x') \, dx', \quad x \in \Gamma,$$

it is the operator S_{-1} in Proposition 2.5. It is also clear from the previous paragraph that

$$\left\langle T_\alpha^m h, h \right\rangle_\Gamma \le 0.$$

This proves the lemma.

We now prove Theorem 2.2 and Theorem 2.4.

Proof (Theorem 2.2). Since T_α^e is bounded, $|a(u, v)| \le C \|u\|_1 \|v\|_1$ for some $C > 0$. By Proposition 2.6, $a(\cdot, \cdot)$ is coercive. Therefore, by the Lax–Milgram theorem, the problem $a(u, v) = b(u, v)$ has a unique solution, $u \in V$ as desired.

Proof (Theorem 2.4). This proof is exactly the same as in the TM case. By the definition of the operator T_α^m, we have

$$\left\langle \frac{\partial w^{n+1}}{\partial y}, T_\alpha^m \frac{\partial w^{n+1}}{\partial y} \right\rangle \le 0.$$

Therefore, it is easy to see that $a(w^{n+1}, w^{n+1}) \ge C \|w^{n+1}\|^2_{H^1(\Omega)}$ for some positive C. Hence, by the Lax–Milgram theorem, the variational problem has a unique solution in V.

3. FINITE ELEMENT APPROXIMATION

Let Ω be covered by a family of triangulations $\mathcal{T}^h$, $0 < h < 1$. Let V^h be the finite dimensional subspace of V consisting of linear functions in Ω,

$$V_h = \{v_h \in V : v_h|_\tau \in P_1 \quad \text{for all } \tau \in \mathcal{T}_h\}.$$

Let $\{\phi_j^h\}_{j=1}^N$ be a linear nodal basis of V_h. Then each $v_h \in V_h$ can be expressed as

$$v_h(x) = \sum_{j=1}^{N} v_j \phi_j^h(x). \tag{3.1}$$

3.1. *TM Polarization*

In this subsection, we discuss the finite element approximation of the TM equation: find u_h^{n+1}, $n = 0, 1, \ldots, \mathcal{N}$, such that

$$\begin{aligned}\left(\nabla u_h^{n+1}, \nabla v_h\right) + \alpha^2\left(\varepsilon_r u_h^{n+1}, v_h\right) - \left\langle T_\alpha^e u_h^{n+1}, v_h\right\rangle_\Gamma &= \alpha^2\left(\varepsilon_r \tilde{u}_h^{n+1}, v_h\right)\\ &\quad + 2\left\langle \frac{\partial u^{inc,n+1}}{\partial y}, v_h\right\rangle_\Gamma, \quad \forall v_h \in V_h.\end{aligned}$$

Substituting (3.1) into the above equation, we obtain the matrix equation

$$(K + \alpha^2 M - P)U^{n+1} = F^{n+1},$$

where the system matrices K, M, P and the right-hand-side vector F^{n+1} are defined as

$$[K]_{ij} = \int_\Omega \nabla\phi_i^h \cdot \nabla\phi_j^h \, dx\, dy, \quad [M]_{ij} = \int_\Omega \varepsilon_r \phi_i^h \phi_j^h \, dx\, dy, \tag{3.2}$$

$$\begin{aligned}[P]_{ij} &= -\frac{\alpha^2}{\pi}\int_\Gamma \phi_j^h(x)\int_\Gamma K_0(\alpha|x - x'|)\phi_i^h(x')\, dx'\, dx\\ &\quad - \frac{1}{\pi}\int_\Gamma \frac{d\phi_j^h}{dx}\int_\Gamma K_0(\alpha|x - x'|)\frac{d\phi_i^h}{dx'}\, dx'\, dx,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\{F^{n+1}\}_j &= \alpha^2\int_\Omega \varepsilon_r \tilde{u}^{n+1}\phi_j^h \, dx\, dy + \int_\Gamma \tilde{H}^{n+1}(x)\phi_j^h(x)\, dx\\ &\quad + 2\int_\Gamma \frac{\partial}{\partial y} u^{i,n+1}(x, 0)\phi_j^h(x)\, dx.\end{aligned} \tag{3.4}$$

The system matrices K and M are standard. We now discuss the computations of the nonlocal boundary matrix P, the right-hand-side vector F^{n+1}, and the scattered field $u^{s,n+1}$ in the upper half plane $\mathcal{U}$ defined in (2.4).

Assume that Γ is divided into segments $\Gamma_j = (x_j, x_{j+1})$ and ϕ_j^h is 1 at x_j and 0 at other nodes. Then the first integral in P becomes

$$\int_\Gamma \phi_j^h(x)\int_\Gamma K_0(\alpha|x - x'|)\phi_i^h(x')\, dx'\, dx = \int_{\Gamma_j} \phi_j^h(x)\int_{\Gamma_i} K_0(\alpha|x - x'|)\phi_i^h(x')\, dx'\, dx. \tag{3.5}$$

When $\Gamma_i \neq \Gamma_j$, $K_0(\alpha|x-x'|)$ is nonsingular. We can integrate (3.5) using a standard interpolatory quadrature. For example, by using the midpoint formula we have

$$\begin{aligned}\int_{\Gamma_j} \phi_j^h(x) \int_{\Gamma_i} K_0(\alpha|x-x'|)\phi_i^h(x')\,dx'\,dx &\approx \frac{|\Gamma_j|}{2}\int_{\Gamma_i} K_0(\alpha|(\xi_j - x'|)\phi_i^h(x')\,dx' \\ &\approx \frac{|\Gamma_j|}{2}\frac{|\Gamma_i|}{2} K_0(\alpha|(\xi_j - \xi_i|),\end{aligned} \tag{3.6}$$

where $|\Gamma_i|$ and $|\Gamma_j|$ are the lengths of Γ_i and Γ_j, respectively, and $\xi_i = \frac{x_{i+1}+x_i}{2}$, $\xi_j = \frac{x_{j+1}+x_j}{2}$.

When $\Gamma_i = \Gamma_j$,

$$\begin{aligned}\int_{\Gamma_i} \phi_i^h(x) \int_{\Gamma_i} K_0(\alpha|x-x'|)\phi_i^h(x')\,dx'\,dx &\approx \frac{|\Gamma_i|}{2}\int_{\Gamma_i} K_0(\alpha|\xi_i - x'|)\phi_i^h(x')\,dx' \\ &\approx \frac{|\Gamma_i|}{2\alpha}\int_0^{\alpha|\Gamma_i|/2} K_0(\tau)\,d\tau.\end{aligned} \tag{3.7}$$

To evaluate the last integral in (3.7) we can use [1, p. 480], for $x > 0$,

$$\begin{aligned}\int_0^x K_0(\tau)\,d\tau = &-\left(\gamma_0 + \ln\frac{x}{2}\right)x\sum_{k=0}^{\infty}\frac{(x/2)^{2k}}{(k!)^2(2k+1)} + x\sum_{k=0}^{\infty}\frac{(x/2)^{2k}}{(k!)^2(2k+1)^2} \\ &+ x\sum_{k=1}^{\infty}\frac{(x/2)^{2k}}{(k!)^2(2k+1)}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right),\end{aligned} \tag{3.8}$$

where γ_0 (Euler's constant) $= 0.5772156649$.

Similarly, we consider the second integral in P,

$$\int_{\Gamma}\frac{d\phi_j^h}{dx}\int_{\Gamma} K_0(\alpha|x-x'|)\frac{d\phi_j^h}{dx'}\,dx'\,dx = \int_{\Gamma_i}\frac{d\phi_j^h}{dx}\int_{\Gamma_j} K_0(\alpha|x-x'|)\frac{d\phi_j^h}{dx'}\,dx'\,dx.$$

When $\Gamma_i \neq \Gamma_j$, we have

$$\int_{\Gamma_j}\frac{d\phi_j^h}{dx}\int_{\Gamma_i} K_0(\alpha|x-x'|)\frac{d\phi_i^h}{dx'}\,dx'\,dx \approx K_0(\alpha|\xi_j - \xi_i|). \tag{3.9}$$

When $\Gamma_i = \Gamma_j$, we have

$$\int_{\Gamma_i}\frac{d\phi_i^h}{dx}\int_{\Gamma_i} K_0(\alpha|x-x'|)\frac{d\phi_i^h}{dx'}\,dx'\,dx \approx \frac{2}{\alpha}\int_0^{\alpha|\Gamma_i|/2} K_0(\tau)\,d\tau. \tag{3.10}$$

Hence, one can use (3.6)–(3.10) to approximate P.

Next, we consider F^{n+1}. For the first integral in F^{n+1}, we notice that in Ω, $\tilde{u}^{n+1}(x,y) \approx \sum_{i=1}^{N}\tilde{u}_i^{n+1}\phi_i^h(x,y)$; thus,

$$\int_{\Omega}\varepsilon_r\tilde{u}^{n+1}\phi_j^h\,dx\,dy \approx \sum_{i=1}^{N}\tilde{u}_i^{n+1}\int_{\Omega}\varepsilon_r\phi_i^h\phi_j^h\,dx\,dy = \sum_{i=1}^{N}[M]_{ij}\tilde{u}_i^{n+1}. \tag{3.11}$$

For the second integral in F^{n+1}, we have

$$\int_\Gamma \tilde{H}^{n+1}\phi_j^h\,dx = \int_{\Gamma_j} \tilde{H}^{n+1}\phi_j^h\,dx \approx \frac{|\Gamma_j|}{2}\tilde{H}^{n+1}(\xi_j), \tag{3.12}$$

where

$$\tilde{H}^{n+1}(\xi_j) = \frac{\alpha^3}{\pi}\int_0^\infty\int_{-\infty}^\infty \frac{y'}{|\boldsymbol{r}'-\xi_j\hat{x}|}K_1(\alpha|\boldsymbol{r}'-\xi_j\hat{x}|)\tilde{u}^{s,n+1}(\boldsymbol{r}')\,d\boldsymbol{r}'.$$

Since the integrand in $\tilde{H}^{n+1}$ decays exponentially with respect to $|\boldsymbol{r}'|$ we can approximate the infinite integral by truncation,

$$\tilde{H}^{n+1}(\xi_j) \approx \frac{\alpha^3}{\pi}\int_0^Y\int_{-X}^X \frac{y'}{|\boldsymbol{r}'-\xi_j\hat{x}|}K_1(\alpha|\boldsymbol{r}'-\xi_j\hat{x}|)\tilde{u}^{s,n+1}(\boldsymbol{r}')\,d\boldsymbol{r}',$$

where $X>0$ and $Y>0$ are sufficiently large. We can subdivide the intervals $[-X, X]$ and $[0, Y]$ into smaller segments X_l and Y_m and form rectangles $R_{lm} = X_l \times Y_m$. For each rectangle R_{lm} denotes its center as λ_{lm}. Then, we further approximate $\tilde{H}^{n+1}$ by

$$\tilde{H}^{n+1}(\xi_j) \approx \frac{\alpha^3}{\pi}\sum_{l=1}^L\sum_{m=1}^M |R_{lm}|\frac{\lambda_{lm}\cdot\hat{y}}{|\lambda_{lm}-\xi_j\hat{x}|}K_1(\alpha|\lambda_{lm}-\xi_j\hat{x}|)\times\tilde{u}^{s,n+1}(\lambda_{lm}), \tag{3.13}$$

where $|R_{lm}|$ is the area of R_{lm}. For this approximation of $\tilde{H}^{n+1}$ the values

$$\tilde{u}^{s,n+1}(\lambda_{lm}) := u^{s,n}(\lambda_{lm}) + \Delta t\dot{u}^{s,n}(\lambda_{lm}) + \frac{\Delta t^2}{2}(1-2\beta)\ddot{u}^{s,n}(\lambda_{lm})$$

need to be precomputed. Thus, let us now consider the computation of the values of the scattered field $u^{s,n}$ at λ_{lm}. By (2.4) we have

$$u^{s,n}(\lambda_{lm}) = \alpha^2\int_0^\infty\int_{-\infty}^\infty G_\alpha^e(\lambda_{lm},\boldsymbol{r}')\tilde{u}^{s,n}(\boldsymbol{r}')\,d\boldsymbol{r}' + \frac{\alpha}{\pi}\int_\Gamma \frac{\lambda_{lm}\cdot\hat{y}}{|\lambda_{lm}-x'\hat{x}|}K_1(\alpha|\lambda_{lm}-x'\hat{x}|)u^n(x',0)\,dx'.$$

Thus, we can estimate $u^{s,n}(\lambda_{lm})$ by

$$u^{s,n}(\lambda_{lm}) \approx \alpha^2\int_0^Y\int_{-X}^X G_\alpha^e(\lambda_{lm},\boldsymbol{r}')\tilde{u}^{s,n}(\boldsymbol{r}')\,d\boldsymbol{r}' + \frac{\alpha}{\pi}\int_\Gamma \frac{\lambda_{lm}\cdot\hat{y}}{|\lambda_{lm}-x'\hat{x}|}K_1(\alpha|\lambda_{lm}-x'\hat{x}|)u^n(x',0)\,dx'.$$

By using the same rectangular subdivision $\{X_i \times Y_j = R_{ij}\}$ of the region $[-X, X]\times[0, Y]$

we can further approximate $u^{s,n}(\lambda_{lm})$ by

$$\begin{aligned}u^{s,n}(\lambda_{lm}) \approx \alpha^2 &\sum_{R_{ij},\lambda_{lm}\notin R_{ij}} |R_{ij}| G^e_\alpha(\lambda_{lm},\lambda_{ij})\tilde{u}^{s,n}(\lambda_{ij}) \\ &+ \alpha^2 \frac{1}{2\pi}\tilde{u}^{s,n}(\lambda_{lm})\left\{|X_l|\int_{Y_m} K_0(\alpha|\lambda_{lm}\cdot\hat{y} - y'|)\,dy' - |R_{lm}|K_0(2\alpha\lambda_{lm}\cdot\hat{y})\right\} \\ &+ \frac{\alpha}{\pi}\sum_{j=1}|\Gamma_j|\frac{\lambda_{lm}\cdot\hat{y}}{|\lambda_{lm} - \xi_j\hat{x}|}K_1(\alpha|\lambda_{lm} - \xi_j\hat{x}|)u^n(\xi_j, 0), \end{aligned} \tag{3.14}$$

where λ_{ij} is the center of the rectangle R_{ij} whose area is $|R_{ij}|$ and ξ_j is the midpoint of the segment Γ_j whose length is $|\Gamma_j|$. To treat the singularity in the integral term above, we apply the same technique as in (3.7),

$$\int_{Y_m} K_0(\alpha|\lambda_{lm}\cdot\hat{y} - y'|)\,dy' = \frac{2}{\alpha}\int_0^{\alpha|Y_m|/2} K_0(\tau)\,d\tau, \tag{3.15}$$

which can be computed by (3.8). Therefore, we can approximate the second integral in F^{n+1} (3.12) by using (3.13), (3.14), and (3.15).

Finally, the last term in F^{n+1} can be easily approximated by the midpoint formula; that is,

$$\begin{aligned}2\int_\Gamma \frac{\partial}{\partial y}u^{inc,n+1}(x,0)\phi^h_j\,dx &\approx 2\frac{|\Gamma_j|}{2}\frac{\partial}{\partial y}u^{inc,n+1}(\xi_j,0) \\ &= |\Gamma_j|\frac{\partial}{\partial y}u^{inc,n+1}(\xi_j,0). \end{aligned} \tag{3.16}$$

Hence, combining the approximations (3.12)–(3.16) yields an approximation of the right-hand-side vector F^{n+1}.

We summarize the time-stepping scheme.

1. Form the matrices K, M, P.
   Time loop: for $n = 0, 1, 2\ldots$
2. Compute the predicted values $\tilde{u}^{n+1}$, $\tilde{\ddot{u}}^{n+1}$ in the interior Ω.
3. Compute the predicted values $\tilde{u}^{n+1}$, $\tilde{\ddot{u}}^{n+1}$ in the exterior $\mathcal{U}$.
4. Form the vector F^{n+1}.
5. Solve $(K + \alpha^2 M - P)u^{n+1} = F^{n+1}$ (in Ω).
6. Compute the solution u^{n+1} in the exterior $\mathcal{U}$.
7. Correct $\ddot{u}^{n+1}$ and $\dot{u}^{n+1}$ in Ω.
8. Correct $\ddot{u}^{n+1}$ and $\dot{u}^{n+1}$ in $\mathcal{U}$.

3.2. *TE Polarization*

In solving the TE problem by using the finite element method, it is more convenient to cast the problem in the mixed variational formulation as follows. Denote

$$\psi^{n+1} = \left.\frac{\partial u^{n+1}}{\partial y}\right|_{y=0^+}.$$

By the continuity condition we have

$$\varepsilon_r^{-1} \frac{\partial u^{n+1}}{\partial y}\bigg|_{y=0^-} = \psi^{n+1}. \tag{3.17}$$

Define

$$\Xi = \left\{\chi \in H^{-1/2}(\Gamma) : \exists v \in V \text{such that } \partial v/\partial y = \chi \text{ on } \Gamma\right\}.$$

Then the mixed variational formulation is as follows: find $u^{n+1} \in V$ and $\psi^{n+1} \in \Phi$ such that

$$\begin{cases} \left(\varepsilon_r^{-1}\nabla u^{n+1}, v\right) + \alpha^2(u^{n+1}, v) - \langle \psi^{n+1}, v\rangle_\Gamma = \alpha^2(\tilde{u}^{n+1}, v), & \forall v \in V, \\ \langle u^{n+1}, \chi\rangle_\Gamma - \left\langle T_\alpha^m \psi^{n+1}, \chi\right\rangle_\Gamma = \langle \tilde{H}^{n+1}, \chi\rangle_\Gamma - 2\left\langle T_\alpha^m \dfrac{\partial u^{inc}}{\partial y}, \chi\right\rangle_\Gamma, & \forall \chi \in \Xi. \end{cases} \tag{3.18}$$

Let $\mathcal{T}_h$ be a family of triangulations of Ω with triangles K. Assume that Γ is subdivided into segments Γ_s. Let $V_h \subset V$ and $\Xi_h \subset \Xi$ be the finite dimensional subspaces defined as

$$\begin{aligned} V_h &= \{v_h \in V : v_h|_K \in P_1(K) \quad \text{for each } K \in \mathcal{T}_h\}, \\ \Xi_h &= \{\chi_h \in \Xi : \chi_h|_{\Gamma s} \in P_0(\Gamma_s) \quad \text{for each } \Gamma_s \subset \Gamma\}. \end{aligned}$$

Denote $\{\phi_j^h\}_{j=1}^N \subset V_h$ as the rooftop basis and $\{\zeta_s^h\}_{s=1}^L \subset \Xi_h$ as the pulse basis. Hence, for each $v_h \in V_h$ and for each $\chi_h \in \Xi_h$ we can write

$$v_h(x, y) = \sum_{j=1}^{N} v_j \phi_j^h(x, y), \quad (x, y) \in \Omega, \quad \chi_h(x) = \sum_{s=1}^{L} \chi_x \zeta_s^h(x), \quad x \in \Gamma.$$

We also need the basis functions $\{\eta_s^h\}_{s=1}^L$ on the segments Γ_s for the trace of v^h on Γ_s; that is,

$$v_h(x, 0) = \sum_{s=1}^{L} v_s \eta_s^h(x), \quad \text{where } \eta_s^h(x) \in P_1(\Gamma_s).$$

Thus, (3.18) can be approximated by a system of linear equations

$$\begin{bmatrix} K + \alpha^2 M & -B \\ B^T & -P \end{bmatrix} \begin{bmatrix} u^{n+1} \\ \chi^{n+1} \end{bmatrix} = \begin{bmatrix} \tilde{F}_1^{n+1} \\ \tilde{F}_2^{n+1} \end{bmatrix},$$

where

$$\begin{aligned} [K]_{ij} &= \int_\Omega \varepsilon_r^{-1} \nabla\phi_i^h \cdot \nabla\phi_j^h \, dx\, dy, \quad [M]_{ij} = \int_\Omega \phi_i^h \phi_j^h \, dx\, dy, \\ [B]_{sj} &= \int_{\Gamma_s} \eta_s^h \, dx, \quad [P]_{st} = -\frac{1}{\pi} \int_{\Gamma_s} \int_{\Gamma_t} K_0(\alpha|x - x'|)\, dx'\, dx, \end{aligned}$$

and

$$\left\{\tilde{F}_1^{n+1}\right\}_j = \alpha^2 \int_\Omega \tilde{u}^{n+1}\phi_j \, dx\, dy, \quad \left\{\tilde{F}_2^{n+1}\right\}_s = 2\int_{\Gamma_s} u^{inc,n+1}(x,0)\, dx + \int_{\Gamma_s} \tilde{H}^{n+1}\, dx,$$

for $i, j = 1, 2, \ldots, N$ and $s, t = 1, 2, \ldots, L$. Here, K, M are $N \times N$, B $N \times L$, and P $M \times M$ matrices respectively.

As before, we denote x_s and x_t as the midpoints of the segments Γ_s, Γ_t, respectively, for $s, t = 1, 2, \ldots, L$, and $x_{s,r}$ and $x_{s,l}$ are the endpoints of the segment Γ_s. The finite element matrices K and M are standard. The matrices B and P can be approximated by

$$B_{j,s} \approx \frac{1}{2}|\Gamma_s|,$$

$$P_{st} \approx \begin{cases} -\dfrac{1}{\pi}|\Gamma_s||\Gamma_t| K_0(\alpha|x_s - x_t|), & \text{if } s \neq t, \\ \dfrac{1}{\pi}\dfrac{|\Gamma_s|}{2\alpha}\displaystyle\int_0^{\alpha|\Gamma_s|/2} K_0(\tau)\, d\tau, & \text{if } s = t. \end{cases}$$

The vectors $\tilde{F}_1^{n+1}$ and $\tilde{F}_2^{n+1}$ are approximated as follows. First, we note that $\tilde{u}^{n+1} \approx \sum_{j=1}^N \tilde{u}_j^{n+1}\phi_j(x, y)$; hence,

$$\left\{\tilde{F}_1^{n+1}\right\}_j \approx \alpha^2 \sum_{j=1}^{N} [M]_{ij}\{\tilde{u}^{n+1}\}_j. \tag{3.19}$$

Next, the vector $\tilde{F}_2^{n+1}$ is approximated by

$$\left\{\tilde{F}_2^{n+1}\right\}_s \approx 2|\Gamma_s| u^{inc,n+1}(x_s, 0) + |\Gamma_s|\tilde{H}^{n+1}(x_s), \tag{3.20}$$

where

$$\tilde{H}^{n+1}(\xi_s) \approx \frac{\alpha^2}{\pi}\sum_{l=1}^{N}\sum_{m=1}^{N} |R_{lm}| K_0(\alpha|\xi_s\hat{x} - \lambda_{lm}|)\tilde{z}^{n+1}(\lambda_{lm}), \tag{3.21}$$

$$\begin{aligned} z^{n+1}(\lambda_{lm}) &= u^{n+1}(\lambda_{lm}) - u^{inc,n+1}(\lambda_{lm}) + u^{ref,n+1}(\lambda_{lm}) \\ &\approx \frac{\alpha^2}{2\pi}\sum_{l'=1}^{N}\sum_{m'=1}^{N} |R_{l'm'}|\{K_0(\alpha|\lambda_{lm} - \lambda_{l'm'}|) + K_0(\alpha|\lambda_{lm} - \lambda^*_{l'm'}|)\}\tilde{z}^{n+1}(\lambda_{l'm'}) \\ &\quad -\frac{1}{\pi}\sum_{s=1}^{L} |\Gamma_s| K_0(\alpha|\lambda_{lm} - x_s\hat{x}|)\chi^{n+1}(x_s). \end{aligned} \tag{3.22}$$

4. NUMERICAL RESULTS

For numerical experiments, let Ω be the rectangular cavity of dimension $1\text{m} \times 0.25$ m, as in Fig. 2. We consider two types of excitations: continuous wave and Gaussian pulse. The cavity is covered by a uniform mesh of triangles so that there are 20 nodes on the longer

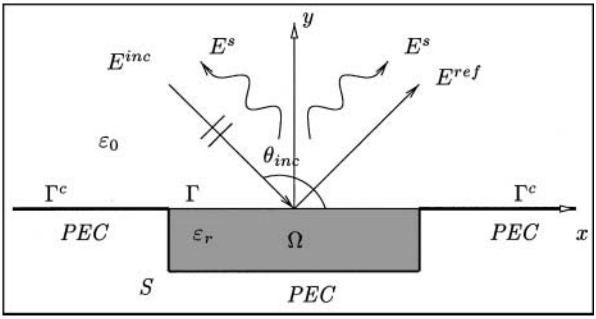


FIG. 2. Rectangular cavity Ω of dimension 1 m $\times$ 0.25 m.

sides and five nodes on the shorter ones. We set

$$\Delta t = 1/20, \quad \gamma = 0.9, \quad \beta = 0.25(0.5 + \gamma)^2.$$

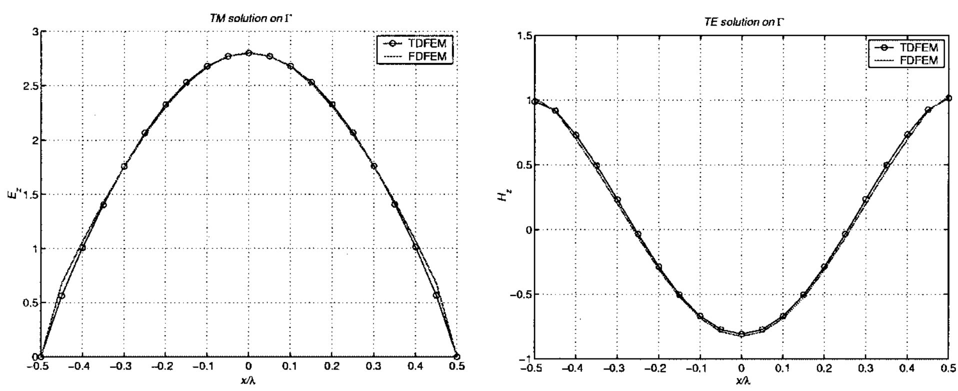
To start the time-marching procedure, we turn on the incident fields at $t = 0$.

4.1. *Incident Continuous Wave*

In this example, the incident field is of the form

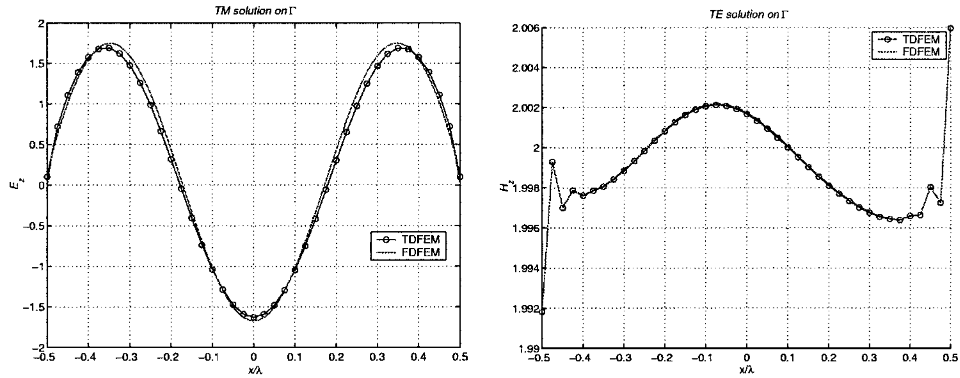
$$u_{inc} = \mathrm{Re}\left\{e^{ik_0(x\cos\theta_{inc} + y\sin\theta_{inc})}e^{ik_0 t}\right\},$$

where $k_0 = 2\pi/\lambda$ is the wave number and $\theta_{inc} = \pi/2$. In the following examples, we use $\lambda = 1$ m (or 300 MHz). After a number of cycles, the steady-state solution follows a basic pattern of the time-harmonic excitation. In Fig. 3, the cavity is empty ($\varepsilon_r = 1$) and TM and TE solutions obtained using the time-domain finite element method (TDFEM) on the cavity opening Γ are plotted and compared to the solutions obtained using the frequency-domain finite element method (FDFEM). The results agree well. In Fig. 4, the cavity is filled with $\varepsilon_r = 4$. Solutions obtained using TDFEM also agree well with those obtained using FDFEM.



(a) TM solution (b) TE solution

FIG. 3. Solutions on Γ at 300 MHz for $\varepsilon_r = 1$.



(a) TM solution (b) TE solution

FIG. 4. Solutions on Γ at 300 MHz for $\varepsilon_r = 4$.

4.2. *Incident Gaussian Pulse*

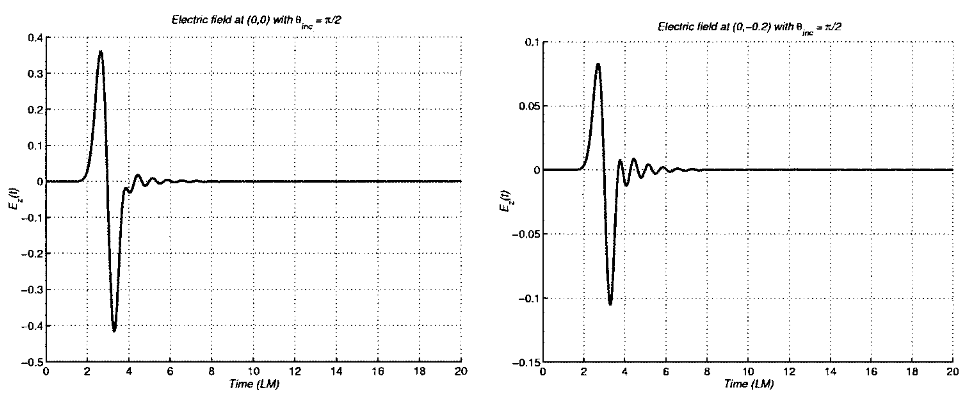
We consider the Gaussian pulse described by (4.1)

$$u_{inc}(x, y, t) = A \frac{4}{T\sqrt{\pi}} e^{-\tau^2},$$

where

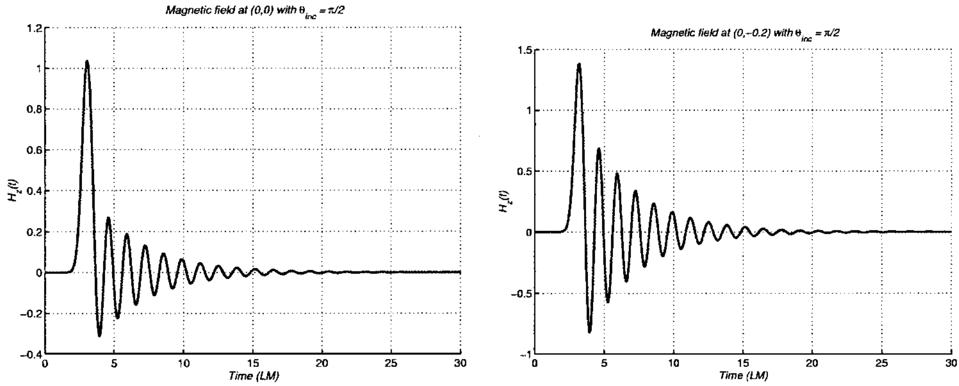
$$\tau = \frac{4(t - t_0 + x \cos\theta_{inc} + y \sin\theta_{inc})}{T}, \quad \theta_{inc} \in [\pi/2, \pi].$$

In the following numerical experiments, we set $A = 1$, $\theta_{inc} = \pi/2$, $T = 2$, and $t_0 = 3$. This means that the Gaussian pulse will reach its maximum at the origin (0, 0) at $t_0 = 3$. Two types of cavities are used: empty (Figs. 5 and 6) and filled with $\varepsilon_r = 2$ (Figs. 7 and 8). In each case, we plot TM and TE solutions at the center of the cavity opening (0, 0) and at the interior point (0, −0.2). Solutions oscillate in the early time and then exponentially decay,



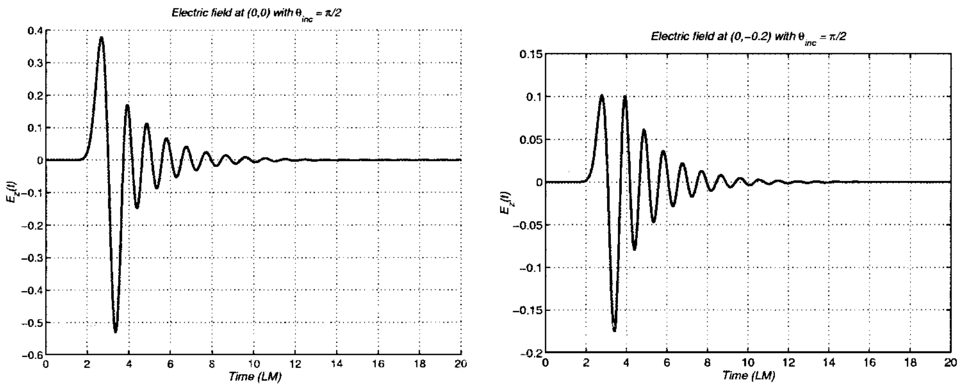
(a) TM solution at (0,0) (b) TM solution at (0,-0.2)

FIG. 5. Empty cavity ($\varepsilon_r = 1$).



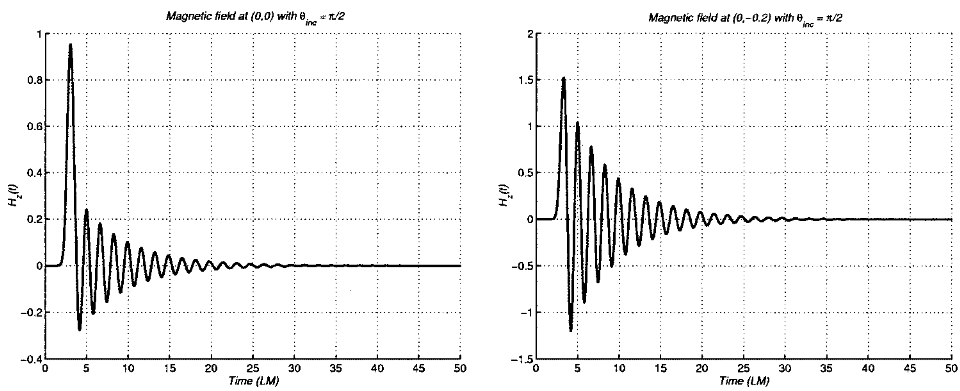
(a) TE solution at (0,0)
(b) TE solution at (0,-0.2)

FIG. 6. Empty cavity ($\varepsilon_r = 1$).



(a) TM solution at (0,0)
(b) TM solution at (0,-0.2)

FIG. 7. Filled cavity with $\varepsilon_r = 2.0$.



(a) TE solution at (0,0)
(b) TE solution at (0,-0.2)

FIG. 8. Filled cavity with $\varepsilon_r = 2.0$.

as expected. The stability of the solutions is clearly seen from the figures. *LM* in the plots denotes a light-meter; i.e., the amount of time for light to travel 1 m in free space.

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